THE TRANSMISSION LINE AS AN IMPEDANCE TRANSFORMER

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Licensed since 1959, G3NXC has operated on all bands from 1·8 to 432MHz using a variety of homebrew and commercial equipment. Currently he is building for the 50MHz band. Professionally he manages an electronics design laboratory in the aerospace industry, and lists among his other outside interests listening to music (preferably written before the 18th century) and photography.

Introduction

In the design of radio frequency systems there is often a need to transform the value of an impedance so that it becomes more usable. As an example, the input impedance of an ideal quarter-wave vertical is in the region of 35Ω. If such an antenna is to be fed with 50Ω cable, its impedance needs to be changed so that the line is terminated correctly. Such transformations may also be required to change, for instance, the input impedance of an amplifier stage so that it can present the correct load to the driver.

Usually impedance transformation is carried out by means of discrete component circuits using inductors and capacitors, the well known pi-tank circuit being an example of such an arrangement. An alternative method is to use a length of transmission line, and this article concentrates on describing such a use of lines.

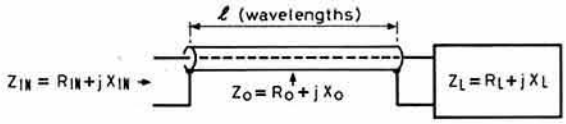

Fig 1. Definition of terms used in the article

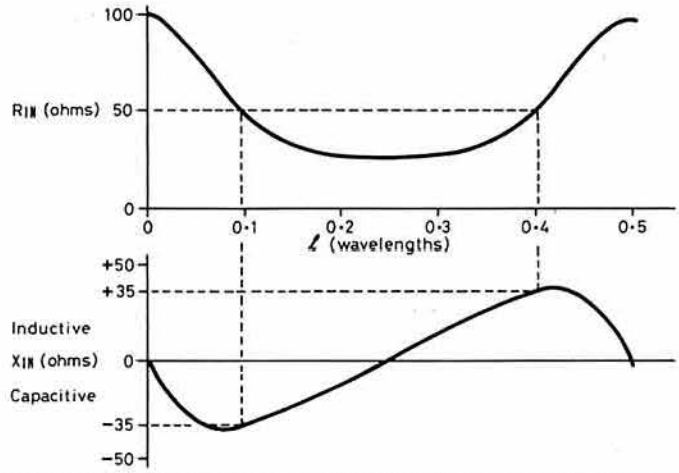

Fig 2. Variation of input impedance with line length, represented as effective series elements. ZL = 100 + j0, Zo = 50 + j0

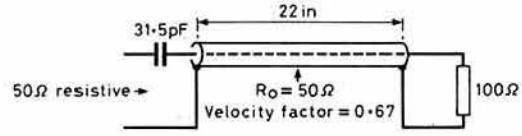

Fig 3. Simple arrangement to transform a 100Ω resistive load to 50Ω resistive

Principle of operation

Consider Fig 1. If the load impedance, ZL, equals the line's characteristic impedance, ZO, the input impedance, ZIN, also equals ZO and is independent of line length. Should ZL not equal ZO, ZIN varies as a function of the line length. In Fig 1 ZO is shown as being a complex impedance made up of a resistance, RO, in series with a reactance, XO. For all practical purposes the reactive part can be considered to be zero so that ZO = RO.

Fig 2 shows the variation of RIN and XIN with line length for the case when a 50Ω line is terminated with a 100Ω resistor. As can be seen, the pattern repeats every half-wave, so it is only necessary to consider line lengths between 0 and 0·5. It should be noted that the length in question is the electrical length, which is the physical length divided by the line's velocity factor.

Of note in Fig 2 is that there are two values of line length where RIN equals 50Ω. In one case RIN is in series with a capacitive reactance of 35Ω, and in the other with an inductive reactance also of 35Ω. By choosing the 0·402 wavelength case and putting a capacitor of appropriate value in series with the input, an effective 50Ω resistive input impedance has been generated. Fig 3 shows the resultant configuration for 144·5MHz.

For every series combination of resistance and reactance there is an equivalent parallel combination. If the equivalent parallel values of RIN and XIN are taken, the variation of these with line length for the same case as above is as shown in Fig 4. As before, there are two lengths where the resistive portion of ZIN is 50Ω, but this time these resistances are in parallel with reactances of 70Ω capacitive or inductive. By using the appropriate length of line and putting a reactance of equal value but opposite type in parallel with the input, an input impedance of 50Ω resistive can be generated.

The most convenient means of generating the appropriate reactance to put in parallel with the input is to use another length of line. A short-circuit line shorter than a quarter-wave looks like a pure inductance, while an open-circuit line of similar length produces a capacitance. For practical reasons a short-circuit line offers the best approach, it is easier to produce —and maintain—a short-circuit than an open-circuit. Fig 5 shows the resultant configuration; many readers will recognize this as being the familiar stub matching arrangement.

The examples given show that transmission line impedance transformers are practical but, unless one is fortunate in wanting to transform 100Ω resistive to 50Ω, they do not help with the design of an appropriate arrangement. In order to carry out the design process it is necessary to delve into the mathematics associated with transmission lines.

Basic line equations

Transmission lines are essentially simple things, all they have to do is to conduct electrical energy from one place to another. Unfortunately, since nature seems at times to have a dislike of simplicity, the mathematics needed to describe the way in which a line performs its simple task can be

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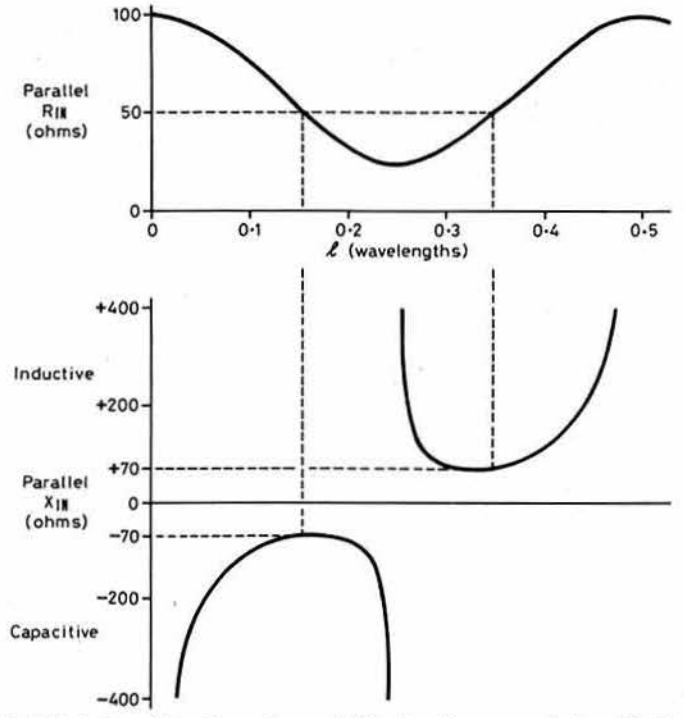

Fig 4. Variation of input impedance with line length, represented as effective parallel elements ZL = 100 + j0, Z_o = 50 + j0

rather overwhelming. I have attempted to minimize the use of complicated mathematics and to break down what remains into easily digested bites. Anyone with a scientific calculator or a home computer should have no problems in performing the calculations.

The basic expression which relates input impedance to line length is:

$$ZIN = RIN + j\,XIN = RO\,\frac{ZL\cos 2\pi l + j\,RO\sin 2\pi l}{RO\cos 2\pi l + j\,ZL\sin 2\pi l} \quad \text{----------(1)}$$

Where l is the line length in electrical wavelengths.

Using the standard trig relationship $\tan 2\pi l = \frac{\sin 2\pi l}{\cos 2\pi l}$ and, to avoid a lot of writing, letting $\tan 2\pi l = \phi$, the above expression can be rewritten as:

$$ZIN = RO\,\frac{ZL + j\,RO\phi}{RO + j\,ZL\phi} \quad \text{----------(2)}$$

It should be noted that, in the above expressions, RIN and XIN are the equivalent series elements (ie a resistor of RINΩ in series with a reactance of XINΩ).

Before continuing, it is worth considering some special cases: the quarter-wave line, the half-wave line and the short and open circuit lines. For the quarter-wave line cos2πl is zero and sin2πl is 1. Putting these values into equation (1) results in the expression:

$$ZIN = \frac{RO^2}{ZL}$$

or:

$$ZIN\,ZL = RO^2$$

With a half-wave line cos 2πl is −1 and sin 2πl is zero so equation (1) reduces to:

$$ZIN = ZL$$

In the case of a short-circuit line ZL = 0, so equation (2) becomes:

$$ZIN = j\,RO\,\phi = j\,RO\tan 2\pi l \quad \text{----------(3)}$$

Similarly for an open-circuit line, ZL is infinite thus:

$$ZIN = \frac{RO}{j\phi} = -j\,RO\cot 2\pi l \quad \text{----------(4)}$$

From (3) and (4) it can be seen that a short-circuit line with a length between 0 and 0·25wavelengths looks like a pure inductive reactance, and with a length between 0·25 and 0·5 wavelengths looks like a pure capacitive reactance. The open-circuit line is the reverse of this.

Returning now to the length calculations. Equation (2) can be manipulated to produce individual expressions for RIN and XIN. From the first of these an equation from which the two values of ϕ (hence l) required to generate the desired value of RIN can be produced. Having determined

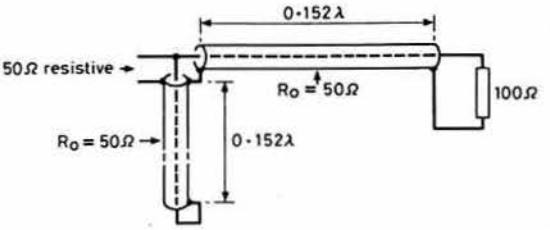

Fig 5. Stub matching arrangement to transform a 100Ω resistive load to 50Ω resistive

the values of ϕ, the expression for XIN can be used to calculate the input reactance. Two sets of equations are required, the first for the case when RIN is in series with XIN, and the second for the parallel case.

Series input calculations

The two expressions required are:

$$RIN = \frac{RO^2\,RL\,(1+\phi^2)}{(RO-XL\phi)^2 + RL^2\phi^2}$$

hence

$$\phi^2(RO^2\,RL - RIN\,XL^2 - RIN\,RL^2) + \phi\,(2RO\,XL\,RIN) + RO^2(RL - RIN) = O \quad \text{----------(5)}$$

and

$$XIN = RO\,\frac{(XL + RO\,\phi)(RO - XL\phi) - RL^2\,\phi}{(RO - XL\,\phi)^2 + RL^2\,\phi^2} \quad \text{---------(6)}$$

Expression (5) appears rather complicated but is actually a standard quadratic equation of the form:

$$a\phi^2 + b\phi + c = O$$

where

$$a = RO^2.RL - RIN.XL^2 - RIN.RL^2$$
$$b = 2.RO.XL.RIN$$
$$c = RO^2.(RL - RIN)$$

The two solutions for ϕ are then given by:

$$\phi 1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \quad \text{----------(7)}$$

and

$$\phi 2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a} \quad \text{----------(8)}$$

By working through the steps of calculating a, b and c, then putting the results into equations (7) and (8), the values of $\phi 1$ and $\phi 2$ can readily be found. The electrical line lengths can be calculated from:

$$l_1 = \frac{\tan^{-1}\phi 1}{2\pi} \quad \text{----------(9)}$$

and

$$l_2 = \frac{\tan^{-1}\phi 2}{2\pi} \quad \text{----------(10)}$$

If the calculator, or computer, being used works in degrees rather than radians when working out trig functions, substitute 360 for the 2π in (9) and (10).

It is likely that l1 or l2 or even both will turn out to be negative values. What this indicates is that the line needs to be shortened by the calculated values—not easy if the line starts off at zero length! Remembering, though, that the impedance pattern repeats every half-wave along a line, the actual length required can be calculated by shortening a half-wave by the appropriate amount. For instance, if l1 is calculated to be −0·1 wavelengths, a line of 0·4 wavelengths will be required.

By putting the values of ϕ found from (7) and (8) in turn into (6), the appropriate values of XIN can be found.

Two things need to be kept in mind when using the above equations:

1. When entering XL it is important that the correct sign is used. XL will be a positive value if inductive, and a negative value if capacitive. XIN, when calculated, follows the same sign convention.

2. The values of RL and XL refer to the series elements needed to describe ZL. If ZL is made up of a resistance and reactance in parallel, the values need to be converted to the equivalent series form. The method for carrying out this conversion is shown later.

Parallel input calculations

The two relevant equations for the parallel input calculations are:

$$RIN = \frac{RL^2 + (XL + RO\phi)^2}{RL\,(1+\phi^2)}$$

hence

$$\phi^2(RO^2 - RIN\,RL) + \phi\,(2XL\,RO) + RL^2 + XL^2 - RL\,RIN = 0 \quad (11)$$

and

$$XIN = RO\frac{RL^2 + (XL + RO\,\phi)^2}{(XL - RO\,\phi)(RO - XL\,\phi) - RL^2\,\phi} \quad \text{------(12)}$$

Equation (11) can be solved using (7) and (8) for which:

$a = RO^2 - RIN.RL$

$b = 2.XL.RO$

$c = RL^2 + XL^2 - RL.RIN$

The line lengths can then be calculated using equations (7) to (10), and the reactances by using equation (12).

General comments

With a 50Ω line, RIN can, in theory, be made 50Ω for any value of ZL. In practical terms, however, it is not sensible to try to cope with very high or very low values of ZL since the accuracy required for the line lengths becomes rather critical. A simple test to find if the arrangement is too critical is to put values of l into equation (1) which are, say, 0·005 wavelengths longer and shorter than the calculated values (this represents about ±0·25in on 144MHz). If RIN departs unacceptably from 50Ω, the arrangement is too critical.

For the case when ZL is high, it may be better to use a 75Ω line but still aiming at RIN equals 50Ω. If dealing with low values of ZL, a low impedance line can be generated by putting higher impedance lines in parallel—eg two equal lengths of 50Ω line in parallel looks like a single 25Ω line.

It is not necessary to restrict the transformation to producing an input resistance which is equal to the RO of the line—any value of RIN can be achieved, within limits. Establishing the actual limits is rather complex but there is a simple test to find out if a particular case is achievable. Before evaluating equations (7) and (8), calculate the term b^2-4ac. If this is positive then the transformation can be achieved, if negative then it cannot.

The techniques are as applicable to balanced lines as they are to coaxial. To a large extent, life is easier with open-wire feeders, since the line can be constructed to have any desired RO simply by choice of wire diameters and spacing. Also with open-wire feeders there are other, more simple, techniques available for carrying out the transformations. For instance, it is possible to calculate the line length needed to make XIN zero and then use a quarter-wave line of appropriate RO to produce the required RIN.

It may be easier to understand the methods described above by giving some examples. The results of these might also be useful to check computer programs written to solve the equations.

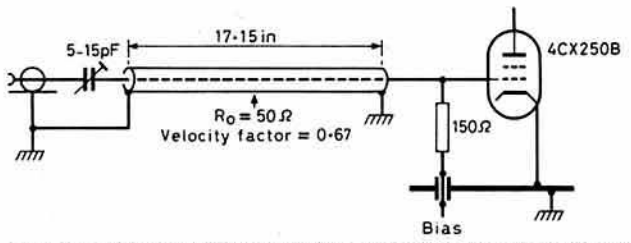

Fig 6. Input transformer for a 4CX250B linear operating on 144MHz. (Note that the actual input impedance of the 4CX250B has been simplified for this example. A practical design would need a better representation of the grid circuit impedance.)

Examples

SERIES ARRANGEMENT

A 4CX250B linear for the 144MHz band is to use a passive grid arrangement with a 150Ω resistor in the grid circuit. The input capacitance of the valve, including strays, is 20pF, and the input is to be matched to a 50Ω line. A design centre frequency of 145·0MHz is required. Since a series capacitor will be required to block the dc bias, it would seem sensible to use this also to tune out the residual reactance of a series arrangement.

At the design centre frequency the 20pF capacitance represents a reactance of 54·88Ω, this being in parallel with the 150Ω resistor. It is necessary first to convert this parallel combination to the equivalent series arrangement. The conversion is achieved by using the standard expressions:

$$RS = \frac{RP\,XP^2}{RP^2 + XP^2}$$

and

$$XS = \frac{RP^2\,XP}{RP^2 + XP^2}$$

Where RS and XS are the series values and RP and XP are the equivalent parallel values.

In this case

$RL = RS = 17{\cdot}71\Omega$

and

$XL = XS = 48{\cdot}40\Omega$

Equation (5) must now be solved remembering that XL is capacitive, and therefore needs to be entered as a negative value. So:

$a = 50^2 \times 17{\cdot}71 - 50 \times (-48{\cdot}4)^2 - 50 \times 17{\cdot}71^2 = -88535{\cdot}205$

$b = 2 \times 50 \times (-48{\cdot}4) \times 50 = -242000{\cdot}000$

$c = 50^2 \times (17{\cdot}71 - 50) = -80725{\cdot}000$

Then, from (7) and (8)

$\phi 1 = -2{\cdot}344467005$ hence $l1 = -0{\cdot}1858333288$

and

$\phi 2 = -0{\cdot}3889089444$ hence $l2 = -0{\cdot}0590319546$

Since both l1 and l2 turn out to be negative values, the actual line lengths required are:

$l1 = 0{\cdot}3141666712$

and

$l2 = 0{\cdot}4409680454$

The reactances can now be calculated by feeding the appropriate values of ϕ into equation (6), remembering to observe the signs of ϕ and XL. In this case the two values are:

$XIN1 = 97{\cdot}76156621\Omega$ at l1

and

$XIN2 = -97{\cdot}76156621\Omega$ at l2

Some readers may raise their eyebrows at the number of decimal places and think that the results represent a rather spurious accuracy. This is, to some extent, true, but it is always better to work to the highest resolution of the computer/calculator during the intermediate calculations then round to an appropriate degree of accuracy at the end of the operations. This minimizes the possibility of a build-up of rounding errors.

The intention is to tune out the residual reactance with a capacitor so l1 is the length required since it yields an inductance in series with RIN. At 145·0MHz the capacitance required to produce a reactance of 97·76Ω is 11·23pF—a 15pF variable would be ideal.

The physical line length can be calculated from l1 by reference to the design centre frequency and the line's velocity factor. Assuming the value of the latter to be 0·67, the length required is 0·436m or 17·15in. The final arrangement would then be as shown in Fig 6.

It must be added that for a real application it would be necessary to take into account all strays in the grid circuit, including the inductance of the connecting leads, in order to establish the actual load impedance.

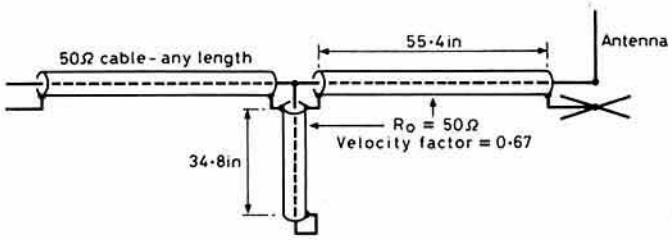

Fig 7. Stub matching arrangement to transform an antenna impedance of 100 + j50Ω to 50Ω resistive on the 28MHz band

PARALLEL ARRANGEMENT

A 28MHz antenna has an input impedance of 100 + j50Ω at 28·4MHz, the design centre frequency, and is to be fed via a 50Ω line.

This time the load impedance is already in series format and needs no conversion, so the first operation is to solve equation (11).

$a = 50^2 - 50 \times 100 = -2{,}500$

$b = 2 \times 50 \times 50 = 5{,}000$

$c = 100^2 + 50^2 - 100 \times 50 = 7{,}500$

So, from equations (7) and (8):

$\phi 1 = -1$ hence $l1 = -0{\cdot}125$ or $l1 = 0{\cdot}375$

$\phi 2 = 3$ hence $l2 = 0{\cdot}1987918088$

The input reactances associated with these two lengths can be calculated from equation (12) and are:

$XIN1 = 50\Omega$ at l1

and

$XIN2 = -50\Omega$ at l2

A short-circuit stub offers an inductive impedance for lengths less than quarter-wave, so the capacitive input case, l2, is chosen. Equation (3) is then used to calculate the length of stub required:

$$l = \frac{\tan^{-1}\frac{X}{50}}{2\pi} = \frac{\tan^{-1}\frac{50}{50}}{2\pi} = 0{\cdot}125$$

At 28·4MHz, assuming a line velocity factor of 0·67, the two line lengths are 55·4 and 34·8in, so the final arrangement would be as shown in Fig 7. □